Determinants

Last Time: Computational Introduction to Determinants.

↳ Cofactor Expansion Formula
(AKA Laplace Expansion Formula)

↳ Many Examples...

↳ Determinants of Elementary Matrices. *

Recall: $\det(P_{i,j}) = -1 \quad (i \neq j)$

$\det(M_i(k)) = k$

$\det(A_{i,j}(k)) = 1$

Def^n: The $n \times n$ determinant function is the function $\det : M_{n\times n} \to \mathbb{R}$ satisfying these conditions:

① $\det(\ell_1, \ell_2, \cdots, k\ell_i + \ell_j, \cdots, \ell_n) = \det(\ell_1, \ell_2, \cdots, \ell_n)$.

② $\det(\ell_1, \ell_2, \cdots, \ell_{i-1}, \ell_j, \ell_{i+1}, \cdots, \ell_{j-1}, \ell_i, \ell_{j+1}, \cdots, \ell_n)$
$= -\det(\ell_1, \ell_2, \cdots, \ell_n)$.

③ $\det(\ell_1, \ell_2, \cdots, k\ell_i, \cdots, \ell_n) = k\det(\ell_1, \cdots, \ell_n)$

④ $\det(I_n) = 1$.

NB: The above properties are indeed satisfied by the Cofactor Expansion formula... (they're a bit nasty to prove..)

Point: determinants are computable using row operations :)

Ex: Compute $\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 5 & 10 & 15 & 20 \end{bmatrix}$.

Sol:

$$\det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 5 & 10 & 15 & 20 \end{bmatrix}$$

$$= 5 \det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 3 & 0 & 1 & -5 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

subtracting multiples of ρ_1 from ρ_2, ρ_3, ρ_4

$$= 5 \det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 0 & 4 & -14 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

swap

$$= 5\cdot(-1) \det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -14 \\ 0 & 2 & 4 & 1 \end{bmatrix}$$

$$= -5 \det\begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 4 & -14 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

row echelon form, "eliminate upwards"

$$= -5 \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

apply scaling rule multiple times..

$$= -5\,(2)(4)(-1)\det(I_4) = -5\cdot 2\cdot 4\cdot -1\cdot 1 = \underline{40}$$ $\boxtimes$

Exercise: Compute $\det(M)$ above via cofactor expansion...

Ex: Compute $\det \begin{bmatrix} -1 & 1 & 5 \\ 5 & 8 & 6 \\ 3 & 9 & -7 \end{bmatrix}$.

Sol: $\det \begin{bmatrix} -1 & 1 & 5 \\ 5 & 8 & 6 \\ 3 & 9 & -7 \end{bmatrix} = \det \begin{bmatrix} -1 & 1 & 5 \\ 0 & 13 & 31 \\ 0 & 12 & 8 \end{bmatrix} \leftarrow$

$= 4 \det \begin{bmatrix} -1 & 1 & 5 \\ 0 & 13 & 31 \\ 0 & 3 & 2 \end{bmatrix}$

$= 4 \cdot \frac{1}{3} \det \begin{bmatrix} -1 & 1 & 5 \\ 0 & \boxed{39} & 93 \\ 0 & 3 & 2 \end{bmatrix} \begin{matrix} \\ \leftarrow \\ \leftarrow \end{matrix}$

Echelon form.

$= \frac{4}{3} \det \begin{bmatrix} -1 & 1 & 5 \\ 0 & \boxed{0} & 67 \\ 0 & 3 & 2 \end{bmatrix} = -\frac{4}{3} \det \begin{bmatrix} -1 & 1 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 67 \end{bmatrix}$

$= -\frac{4}{3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 67 \end{bmatrix} = -\frac{4}{3} (-1)(3)(67) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= -\frac{4}{3} (-1)(3)(67) \cdot 1 = 268$ ✓ ☒

Sol 2 (Via Cofactor Expansion):

$\det \begin{bmatrix} \overset{+}{-1} & \overset{-}{1} & \overset{+}{5} \\ 5 & 8 & 6 \\ 3 & 9 & -7 \end{bmatrix} = -\det \begin{bmatrix} 8 & 6 \\ 9 & -7 \end{bmatrix} - \det \begin{bmatrix} 5 & 6 \\ 3 & -7 \end{bmatrix} + 5 \det \begin{bmatrix} 5 & 8 \\ 3 & 9 \end{bmatrix}$

$= -(8 \cdot -7 - 6 \cdot 9) - (5 \cdot -7 - 6 \cdot 3) + 5(5 \cdot 9 - 3 \cdot 8)$

$= -(-56 - 54) - (-35 - 18) + 5(45 - 24)$

$= 110 + 53 + 5(21) = 163 + 105 = 268$ ☒

Prop: The cofactor Expansion Formula and the properties of det given at the beginning of the lecture determine the same quantity for every $n \times n$ matrix. In particular, the determinant function is given by either.

Ex: Compute $\det \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \\ 0 & 3 & 4 \end{bmatrix}$

Sol:

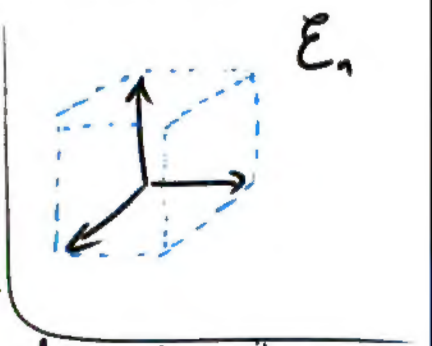
$$\det \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 3 \\ 0 & 3 & 4 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{bmatrix}$$

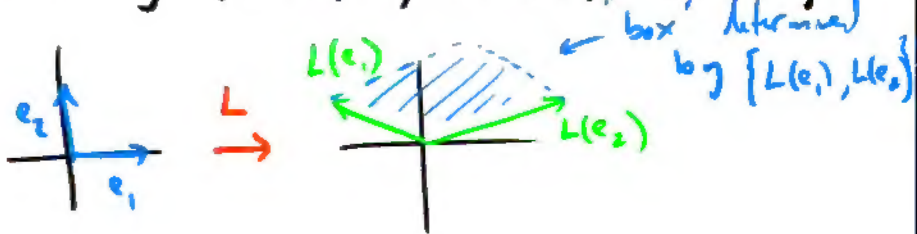
$$= \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\square$

Prop: Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let $[L]$ be the matrix of L with respect to the standard basis on $\mathbb{R}^n$ (i.e. $[L] = [L(e_1) \mid L(e_2) \mid \cdots \mid L(e_n)]$). The determinant $\det [L]$ is the "signed volume" of the "box" determined by $\{L(e_1), L(e_2), \ldots, L(e_n)\}$.

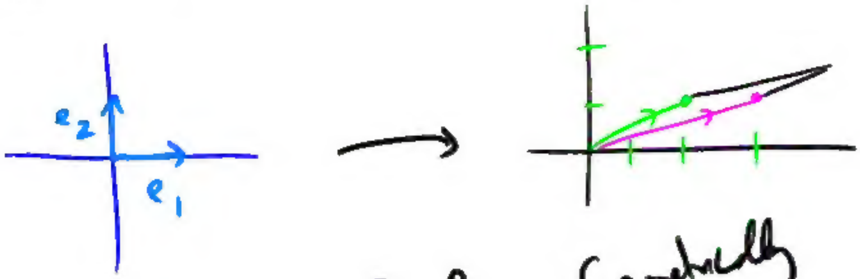


Picture in $\mathbb{R}^2$:

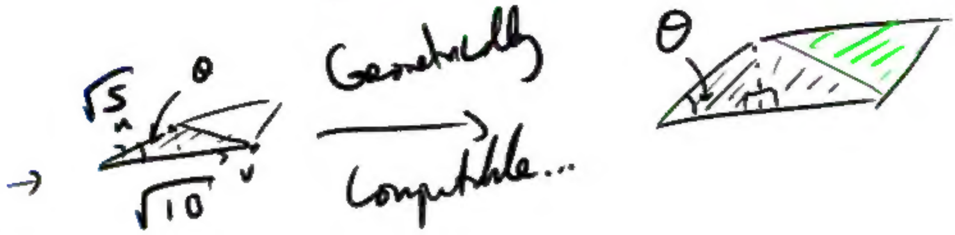


NB: Proof omitted for time, see Hefferon... $\boxed{\diagup\!\!/}$

Ex: Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ have matrix $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.



"Box" = "parallelopiped".



$\det \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = 2-3 = -1 \leftarrow \therefore$ Area $= |-1| = 1$. $\boxed{\diagup\!\!/}$

Cor: The determinant is multiplicative. I.E.

For $A, B \in M_{n\times n}$ we have $\det(AB) = \det(A)\det(B)$.

Pf: A and B determine two linear transformations $\mathbb{R}^n \to \mathbb{R}^n$. The product is the matrix of their composition. Then $\det(AB) \overset{*}{=}$ volume of the parallelopiped determined by $AB(\mathcal{E}_n) = A(B\mathcal{E}_n)$. So we see

$\det(AB) \overset{*}{=} \det(A) \cdot$ volume (parallelopiped given by $B\mathcal{E}_n$)

$\overset{*}{=} \det(A)\det(B)$. :) $\boxed{\diagup\!\!/}$

Proposition :)

NB: This isn't particularly surprising... The definition of the determinant given today encodes the conditions

$\det$ (Product of elem mats) = prod (dets of the elem mats) :)

Cor: Suppose A is invertible. Then $\det(A^{-1}) = \frac{1}{\det(A)}$.

pf: If A is invertible, then $I_n = A^{-1}A$,

so $1 = \det(I_n) = \det(A^{-1}A) = \det(A^{-1}) \cdot \det(A)$.

Hence dividing both sides by $\det(A)$ yields result. $\boxtimes$

Exercise: Check for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ directly ...

Cor: Let A be an $n \times n$ matrix. Then $\det(A) \neq 0$ if and only if A is invertible.

pf: If A is invertible, $\det(A^{-1}) \cdot \det(A) \neq 0$, so $\det(A) \neq 0$.

If $\det(A) \neq 0$, then $L_A : \mathbb{R}^n \to \mathbb{R}^n$ determined by A takes the parallelopiped of $\mathcal{E}_n$ to a parallelopiped of nonzero volume. Moreover, if $L_A(x) = 0$ for $x \neq 0$, then extending $\{x\}$ to a basis of $\mathbb{R}^n$ would yield a parallelopiped which maps under L_A to a zero-volume parallelopiped, hence contradicting the theorem. $\boxtimes$